Risk Assessment and Management Decisions

www.ramd.reapress.com

Risk Assess. Manage. Decis. Vol. 2, No. 4 (2025) 254-265.

Paper Type: Original Article

Fuzzy and Neutrosophic Multi-Criteria Risk Management

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Citation:

Received: 18 February 2025 Revised: 21 April 2025 Accepted: 18 May 2025	Fujita, T. (2025). Fuzzy and neutrosophic multi-Criteria risk management. Risk Assessment and Management Decisions, 2(4), 254-265.

Abstract

Risk Management is the process of identifying, assessing, and mitigating potential losses to minimize the impact on organizational objectives [1, 2]. Fuzzy risk management uses fuzzy sets to represent vague likelihood and impact, then ranks and mitigates risks under imprecise information. Neutrosophic risk management models each risk with truth, indeterminacy, and falsity degrees, enabling decisions when data are conflicting or incomplete. In this paper, we define Multi-Criteria Risk Management, Fuzzy Multi-Criteria Risk Management, and Neutrosophic Multi-Criteria Risk Management, and we investigate their fundamental properties. In future work, we expect that domain experts will evaluate and validate the proposed models and concepts.

Keywords: Fuzzy Multi-Criteria Risk Management, Multi-Criteria Risk Management, Fuzzy Set, Neutrosophic Multi-Criteria Risk Management, Neutrosophic Set

1|Preliminaries

This section gathers the background notions and notation required for the main results.

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https://doi.org/10.48314/ramd.v2i4.74

1.1|Fuzzy Risk Management

Risk Management is the process of identifying, assessing, and mitigating potential losses to minimize the impact on organizational objectives[1, 2]. Fuzzy risk management uses fuzzy sets to represent vague likelihood and impact, then ranks and mitigates risks under imprecise information[3]. The integration of fuzzy logic with risk management has been extensively examined in various research studies [4, 5].

Definition 1.1 (Fuzzy Set)**.** [6] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \le \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 1.2 (Mathematical Framework for Fuzzy Risk Management)**.** [3] Let $(\Omega, \mathcal{F}, P)$ be a probability space and $D \subseteq \mathbb{R}^n$ a nonempty closed convex decision set. Let

$$L : D \longrightarrow L^\infty(\Omega, \mathcal{F}, P), \qquad x \mapsto L(x)$$

be the mapping which assigns to each decision $x \in D$ its (essentially bounded) loss random variable $L(x)$. Let

$$\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a (coherent) risk measure as in the crisp framework. Finally, let

$$\Phi : \mathbb{R} \longrightarrow [0,1]$$

be a continuous, strictly decreasing "satisfaction–risk" mapping (e.g. $\Phi(r) = e^{-\lambda r}$ for some $\lambda > 0$). Then we define the *fuzzy decision set*

$$\mathcal{A} = \big\{(x, \mu_{\mathcal{A}}(x)) \mid x \in D\big\},$$

where the membership function $\mu_{\mathcal{A}} : D \to [0,1]$ is

$$\mu_{\mathcal{A}}(x) = \Phi\big(\rho\big(L(x)\big)\big).$$

The *fuzzy risk management problem* is to choose

$$x^* \in \arg\max_{x \in D} \mu_{\mathcal{A}}(x),$$

equivalently $\min_{x \in D} \rho\big(L(x)\big)$ with gradual preference captured by Φ.

1.2|Neutrosophic Risk Management

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, thereby addressing situations that are neither entirely true nor entirely false. This framework provides a more flexible representation of uncertainty and ambiguity [7, 8, 9]. Their definitions are presented below.

Definition 1.3 (Neutrosophic Set)**.** [10] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Neutrosophic risk management models each risk with truth, indeterminacy, and falsity degrees, enabling decisions when data are conflicting or incomplete [3]. The integration of Neutrosophic logic with risk management has been extensively examined in various research studies [11, 12, 13].

Definition 1.4 (Mathematical Framework for Neutrosophic Risk Management)**.** [3] Let $(\Omega, \mathcal{F}, P)$ be a probability space and $D \subseteq \mathbb{R}^n$ a nonempty closed convex decision set. Define the loss mapping

$$L : D \longrightarrow L^\infty(\Omega, \mathcal{F}, P), \quad x \mapsto L(x),$$

and let

$$\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a coherent risk measure. Further, let

$$\Phi_T,\ \Phi_I,\ \Phi_F : \mathbb{R} \longrightarrow [0,1]$$

be continuous functions generating, respectively, the truth-, indeterminacy-, and falsity-membership degrees. Then for each $x \in D$ define

$$T(x) = \Phi_T\big(\rho(L(x))\big), \quad I(x) = \Phi_I\big(\rho(L(x))\big), \quad F(x) = \Phi_F\big(\rho(L(x))\big).$$

The *neutrosophic decision set* is

$$\mathcal{N} = \big\{ \big(x,\, (T(x), I(x), F(x))\big) \mid x \in D \big\}.$$

Given nonnegative weights w_T, w_I, w_F with $w_T + w_I + w_F = 1$, define the aggregated neutrosophic score

$$S(x) = w_T\, T(x) \ +\ w_I\big(1 - I(x)\big) \ +\ w_F\big(1 - F(x)\big).$$

The *neutrosophic risk management problem* is the optimization

$$x^* \ \in\ \arg\max_{x \in D}\ S(x).$$

Example 1.5 (Neutrosophic Risk Management for Two Investment Projects)**.** Consider a firm choosing between two projects

$$D = \{x_A, x_B\},$$

with a two–state market

$$\Omega = \{\omega_{\text{good}}, \omega_{\text{bad}}\}, \qquad P(\omega_{\text{good}}) = 0.6,\ P(\omega_{\text{bad}}) = 0.4.$$

The monetary loss (in million USD) of each project is

$$L(x_A)(\omega_{\text{good}}) = 1, \quad L(x_A)(\omega_{\text{bad}}) = 5,$$
$$L(x_B)(\omega_{\text{good}}) = 2, \quad L(x_B)(\omega_{\text{bad}}) = 3.$$

Take as coherent risk measure the expected loss

$$\rho(Z) := \mathbb{E}[Z].$$

Then

$$\rho(L(x_A)) = 0.6 \cdot 1 + 0.4 \cdot 5 = 2.6, \qquad \rho(L(x_B)) = 0.6 \cdot 2 + 0.4 \cdot 3 = 2.4.$$

Define the neutrosophic generators (using a scale parameter $M = 5$)

$$\Phi_T(r) := 1 - \frac{r}{M}, \qquad \Phi_I(r) := 0.2, \qquad \Phi_F(r) := \frac{r}{M}, \quad (0 \le r \le M).$$

Then the neutrosophic evaluations of x_A and x_B are

$$(T(x_A), I(x_A), F(x_A)) = \big(1 - \tfrac{2.6}{5},\, 0.2,\, \tfrac{2.6}{5}\big) = (0.48,\, 0.20,\, 0.52),$$
$$(T(x_B), I(x_B), F(x_B)) = \big(1 - \tfrac{2.4}{5},\, 0.2,\, \tfrac{2.4}{5}\big) = (0.52,\, 0.20,\, 0.48),$$

so that in both cases $0 \le T + I + F = 1.2 \le 3$, as required for a neutrosophic triple.

Choose aggregation weights

$$w_T = 0.5, \quad w_I = 0.2, \quad w_F = 0.3, \qquad w_T + w_I + w_F = 1.$$

The aggregated neutrosophic scores are

$$S(x_A) = 0.5 \cdot 0.48 + 0.2 \cdot (1 - 0.20) + 0.3 \cdot (1 - 0.52) = 0.544,$$
$$S(x_B) = 0.5 \cdot 0.52 + 0.2 \cdot (1 - 0.20) + 0.3 \cdot (1 - 0.48) = 0.576.$$

Since $S(x_B) > S(x_A)$, the neutrosophic risk management framework selects project x_B as the preferred decision.

2|Main Results

In this section, we present and explain the results of this paper.

2.1|Multi-Criteria Risk Management

Multi-Criteria Risk Management evaluates decisions across multiple risk dimensions (e.g., financial, operational, reputational), balancing trade-offs via Pareto efficiency or weighted aggregation to minimize overall exposure [14, 15, 16, 17].

Definition 2.1 (Multi-Criteria Risk Management Problem)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and $D \subseteq \mathbb{R}^n$ a nonempty set of admissible decisions. Let

$$K := \{1, 2, \ldots, m\}$$

be a finite index set of risk criteria (for example, financial loss, reputational loss, regulatory penalties, and so on).

For each $k \in K$, let

$$L_k : D \longrightarrow L^\infty(\Omega, \mathcal{F}, P)$$

be the loss mapping associated with criterion k, and let

$$\rho_k : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a (coherent) risk measure for criterion k. Define the *criterion-specific risk functional*

$$J_k : D \longrightarrow \mathbb{R}, \qquad J_k(x) := \rho_k\big(L_k(x)\big) \quad (k = 1, \ldots, m),$$

and collect them into the *risk vector*

$$J : D \longrightarrow \mathbb{R}^m, \qquad J(x) := \big(J_1(x), \ldots, J_m(x)\big).$$

(i) A decision $x^* \in D$ is called *Pareto-efficient* (or *Pareto-optimal*) for the multi-criteria risk problem if there is no $x \in D$ such that

$$J_k(x) \le J_k(x^*) \ \text{ for all } k \in K, \quad \text{and} \quad J_{k_0}(x) < J_{k_0}(x^*) \ \text{ for some } k_0 \in K.$$

(ii) Let $w = (w_1, \ldots, w_m)$ be a vector of nonnegative weights with $\sum_{k=1}^m w_k = 1$. The *aggregated risk functional* is

$$R_w : D \longrightarrow \mathbb{R}, \qquad R_w(x) := \sum_{k=1}^m w_k\, J_k(x) = \sum_{k=1}^m w_k\, \rho_k\big(L_k(x)\big).$$

The *(scalarized) Multi-Criteria Risk Management problem* with weights w is

$$\min_{x \in D} R_w(x).$$

We call the tuple

$$\mathsf{MCRM} := \big(D, \{L_k\}_{k=1}^m, \{\rho_k\}_{k=1}^m\big)$$

a *Multi-Criteria Risk Management structure.*

Remark 2.2. Intuitively, Risk Management corresponds to optimizing a single scalar risk $J(x)$, while Multi-Criteria Risk Management simultaneously considers several risk components $J_1(x), \ldots, J_m(x)$ and either (a) searches for Pareto-efficient decisions, or (b) aggregates them by a weight vector w.

Example 2.3 (Multi-Criteria Risk Management for Two IT Outsourcing Contracts)**.** Consider two candidate outsourcing contracts

$$D = \{x_A, x_B\},$$

evaluated under two risk criteria:

$$k = 1 : \ \text{financial loss (in million USD)}, \qquad k = 2 : \ \text{reputational damage score.}$$

Assume a two–state environment

$$\Omega = \{\omega_{\text{normal}}, \omega_{\text{incident}}\}, \quad P(\omega_{\text{normal}}) = 0.8,\ P(\omega_{\text{incident}}) = 0.2.$$

Financial loss L_1 and reputational loss L_2 are given by

$$\begin{aligned} L_1(x_A)(\omega_{\text{normal}}) &= 1, \quad L_1(x_A)(\omega_{\text{incident}}) = 5, \\ L_1(x_B)(\omega_{\text{normal}}) &= 2, \quad L_1(x_B)(\omega_{\text{incident}}) = 3, \\ L_2(x_A)(\omega_{\text{normal}}) &= 1, \quad L_2(x_A)(\omega_{\text{incident}}) = 4, \\ L_2(x_B)(\omega_{\text{normal}}) &= 1, \quad L_2(x_B)(\omega_{\text{incident}}) = 2. \end{aligned}$$

For each criterion $k = 1, 2$ we take the coherent risk measure

$$\rho_k(Z) := \mathbb{E}[Z],$$

so that

$$\begin{aligned} J_1(x_A) &= \rho_1(L_1(x_A)) = 0.8 \cdot 1 + 0.2 \cdot 5 = 1.8, \qquad J_1(x_B) = 0.8 \cdot 2 + 0.2 \cdot 3 = 2.2, \\ J_2(x_A) &= \rho_2(L_2(x_A)) = 0.8 \cdot 1 + 0.2 \cdot 4 = 1.6, \qquad J_2(x_B) = 0.8 \cdot 1 + 0.2 \cdot 2 = 1.2. \end{aligned}$$

Thus the risk vectors are

$$J(x_A) = (1.8, 1.6), \qquad J(x_B) = (2.2, 1.2).$$

With weights $w = (w_1, w_2) = (0.6, 0.4)$, the aggregated risks are

$$R_w(x_A) = 0.6 \cdot 1.8 + 0.4 \cdot 1.6 = 1.72,$$

$$R_w(x_B) = 0.6 \cdot 2.2 + 0.4 \cdot 1.2 = 1.80.$$

Since $R_w(x_A) < R_w(x_B)$, the scalarized Multi-Criteria Risk Management problem selects x_A as the preferred contract.

Theorem 2.4 (Multi-Criteria Risk Management Generalizes Risk Management)**.** *Let*

$$\mathsf{RM} := (D, L, \rho)$$

be a Risk Management structure in the sense above, with risk functional $J(x) := \rho(L(x))$. Then there exists a Multi-Criteria Risk Management structure

$$\mathsf{MCRM} := \big(D, \{L_k\}_{k=1}^m, \{\rho_k\}_{k=1}^m\big)$$

and a weight vector w such that the Risk Management problem

$$\min_{x \in D} J(x)$$

is exactly the scalarized Multi-Criteria Risk Management problem

$$\min_{x \in D} R_w(x)$$

for MCRM*.*

In particular, every Risk Management problem is a special case of a Multi-Criteria Risk Management problem.

Proof: We construct an explicit embedding of RM into the class of Multi-Criteria Risk Management structures. Set $m := 1$ and let $K := \{1\}$. Define

$$L_1 := L, \qquad \rho_1 := \rho.$$

Then, by Definition 2.1, the corresponding criterion-specific risk functional is

$$J_1(x) := \rho_1\big(L_1(x)\big) = \rho\big(L(x)\big) = J(x) \quad \text{for all } x \in D.$$

The resulting Multi-Criteria Risk Management structure is

$$\mathsf{MCRM} := \big(D, \{L_1\}, \{\rho_1\}\big),$$

with risk vector $J(x) = (J_1(x)) \in \mathbb{R}$.

Now choose the weight vector $w = (1) \in \mathbb{R}^1$, so that the aggregated risk functional R_w is

$$R_w(x) = \sum_{k=1}^{1} w_k\, J_k(x) = 1 \cdot J_1(x) = J(x) = \rho\big(L(x)\big) \quad \text{for all } x \in D.$$

Hence the scalarized Multi-Criteria Risk Management problem

$$\min_{x\in D} R_w(x)$$

is identical, pointwise, to the original Risk Management problem

$$\min_{x\in D} J(x) = \min_{x\in D} \rho\big(L(x)\big).$$

Therefore any optimal decision for the Risk Management problem is also optimal for the corresponding Multi-Criteria problem with $m = 1$ and $w = (1)$, and vice versa. This shows that Risk Management is recovered as a special case of Multi-Criteria Risk Management, so Multi-Criteria Risk Management strictly generalizes the classical single-criterion setting. □

2.2|Fuzzy Multi-Criteria Risk Management

Fuzzy Multi-Criteria Risk Management represents each criterion's risk with graded membership, aggregating imprecise evaluations to prioritize safer, more satisfactory decisions.

Definition 2.5 (Fuzzy Multi-Criteria Risk Management)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $D \subseteq \mathbb{R}^n$ be a nonempty set of admissible decisions. Let

$$K := \{1, \dots, m\}$$

be a finite index set of risk criteria.

For each $k \in K$, let

$$L_k : D \longrightarrow L^\infty(\Omega, \mathcal{F}, P), \qquad x \longmapsto L_k(x)$$

be the (essentially bounded) loss random variable associated with decision x under criterion k, and let

$$\rho_k : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a (coherent) risk measure for criterion k. Define the criterion-wise risk functionals

$$J_k : D \longrightarrow \mathbb{R}, \qquad J_k(x) := \rho_k\big(L_k(x)\big), \quad k = 1, \dots, m.$$

Collecting these, we obtain the risk vector

$$J : D \longrightarrow \mathbb{R}^m, \qquad J(x) := \big(J_1(x), \dots, J_m(x)\big).$$

Let

$$\Psi : \mathbb{R}^m \longrightarrow [0, 1]$$

be a continuous function, nonincreasing in each coordinate (i.e. J_k larger $\Rightarrow \Psi(J)$ not larger), called the *fuzzy multi-criteria satisfaction function*. The *fuzzy membership* of a decision $x \in D$ is

$$\mu_{\mathrm{FM}}(x) := \Psi\big(J(x)\big) = \Psi\big(J_1(x), \dots, J_m(x)\big).$$

The *Fuzzy Multi-Criteria Risk Management problem* is to choose

$$x^* \in \arg\max_{x\in D} \mu_{\mathrm{FM}}(x) = \arg\max_{x\in D} \Psi\big(J_1(x), \dots, J_m(x)\big).$$

The structure

$$\mathsf{FMCRM} := \big(D, \{L_k\}_{k=1}^m, \{\rho_k\}_{k=1}^m, \Psi\big)$$

is called a *Fuzzy Multi-Criteria Risk Management structure*.

Remark 2.6. Intuitively, $J_k(x)$ is the scalar risk under criterion k, and $\Psi(J_1(x), \dots, J_m(x))$ aggregates these into a fuzzy satisfaction degree in $[0, 1]$; decisions with higher $\mu_{\mathrm{FM}}(x)$ are preferred.

Example 2.7 (Fuzzy Multi-Criteria Risk Management for IT Project Selection)**.** Consider two IT projects

$$D = \{x_A, x_B\}$$

and two risk criteria:

$$k = 1: \text{ financial loss (million USD)}, \qquad k = 2: \text{ schedule delay (penalty units)}.$$

Assume a two–state environment

$$\Omega = \{\omega_{\text{normal}}, \omega_{\text{incident}}\}, \quad P(\omega_{\text{normal}}) = 0.8,\ P(\omega_{\text{incident}}) = 0.2.$$

Criterion-wise losses are

$$\begin{aligned} L_1(x_A)(\omega_{\text{normal}}) &= 1, \quad L_1(x_A)(\omega_{\text{incident}}) = 5, \\ L_1(x_B)(\omega_{\text{normal}}) &= 2, \quad L_1(x_B)(\omega_{\text{incident}}) = 3, \\ L_2(x_A)(\omega_{\text{normal}}) &= 1, \quad L_2(x_A)(\omega_{\text{incident}}) = 4, \\ L_2(x_B)(\omega_{\text{normal}}) &= 1, \quad L_2(x_B)(\omega_{\text{incident}}) = 2. \end{aligned}$$

For each $k = 1, 2$ take the coherent risk measure

$$\rho_k(Z) := \mathbb{E}[Z],$$

so the criterion-specific risks are

$$\begin{aligned} J_1(x_A) &= 0.8 \cdot 1 + 0.2 \cdot 5 = 1.8, \qquad J_1(x_B) = 0.8 \cdot 2 + 0.2 \cdot 3 = 2.2, \\ J_2(x_A) &= 0.8 \cdot 1 + 0.2 \cdot 4 = 1.6, \qquad J_2(x_B) = 0.8 \cdot 1 + 0.2 \cdot 2 = 1.2. \end{aligned}$$

Hence

$$J(x_A) = (1.8, 1.6), \qquad J(x_B) = (2.2, 1.2).$$

Let $w = (w_1, w_2) = (0.6, 0.4)$ and define the aggregated (crisp) risk

$$R_w(x) := w_1 J_1(x) + w_2 J_2(x),$$

so that

$$R_w(x_A) = 0.6 \cdot 1.8 + 0.4 \cdot 1.6 = 1.72, \qquad R_w(x_B) = 0.6 \cdot 2.2 + 0.4 \cdot 1.2 = 1.80.$$

To obtain a fuzzy multi-criteria satisfaction degree, choose

$$\Psi(J_1, J_2) := \exp\bigl(-\tfrac{1}{2}(w_1 J_1 + w_2 J_2)\bigr) = \exp\bigl(-\tfrac{1}{2} R_w\bigr),$$

which is continuous and nonincreasing in each coordinate. Then the fuzzy memberships are

$$\mu_{\text{FM}}(x_A) = \exp\bigl(-\tfrac{1}{2} \cdot 1.72\bigr) \approx 0.423, \qquad \mu_{\text{FM}}(x_B) = \exp\bigl(-\tfrac{1}{2} \cdot 1.80\bigr) \approx 0.407.$$

Since $\mu_{\text{FM}}(x_A) > \mu_{\text{FM}}(x_B)$, the Fuzzy Multi-Criteria Risk Management framework selects project x_A as the preferred decision.

Theorem 2.8 (Fuzzy Multi-Criteria Risk Management generalizes both)**.** *The framework of Fuzzy Multi-Criteria Risk Management in Definition 2.5 simultaneously generalizes:*

(i) (Single-criterion) Fuzzy Risk Management, and

(ii) (Crisp) Multi-Criteria Risk Management with scalarization.

Proof: (i) *Fuzzy Risk Management as a special case.* Consider a (single-criterion) fuzzy risk management model: there is a single loss mapping $L : D \to L^\infty(\Omega, \mathcal{F}, P)$, a risk measure $\rho : L^\infty \to \mathbb{R}$, and a decreasing function $\Phi : \mathbb{R} \to [0, 1]$, with fuzzy membership $\mu_{\text{F}}(x) := \Phi(\rho(L(x)))$.

Embed this into Definition 2.5 by taking

$$m = 1, \quad L_1 := L, \quad \rho_1 := \rho,$$

so that

$$J_1(x) = \rho_1(L_1(x)) = \rho(L(x)).$$

Define $\Psi : \mathbb{R}^1 \to [0, 1]$ by

$$\Psi(r) := \Phi(r), \quad r \in \mathbb{R}.$$

Then

$$\mu_{\mathrm{FM}}(x) = \Psi\big(J_1(x)\big) = \Phi\big(\rho(L(x))\big) = \mu_{\mathrm{F}}(x),$$

and hence

$$\arg\max_{x\in D} \mu_{\mathrm{FM}}(x) = \arg\max_{x\in D} \mu_{\mathrm{F}}(x).$$

Thus the classical Fuzzy Risk Management problem is recovered as the case $m = 1$ of Fuzzy Multi-Criteria Risk Management.

(ii) *Multi-Criteria Risk Management as a special case.* Consider a scalarized Multi-Criteria Risk Management model with the same $D, \{L_k\}, \{\rho_k\}$, and a weight vector $w = (w_1, \ldots, w_m)$ with $w_k \geq 0$ and $\sum_{k=1}^{m} w_k = 1$. The scalarized aggregated risk is

$$R_w(x) := \sum_{k=1}^{m} w_k\, J_k(x).$$

The usual multi-criteria problem is

$$\min_{x\in D} R_w(x).$$

Choose a strictly decreasing bijection $\Phi : \mathbb{R} \to [0, 1]$ (for instance $\Phi(r) := e^{-\lambda r}$ for some $\lambda > 0$), and define

$$\Psi(J_1, \ldots, J_m) := \Phi\Big(\sum_{k=1}^{m} w_k\, J_k\Big).$$

Then

$$\mu_{\mathrm{FM}}(x) = \Psi\big(J_1(x), \ldots, J_m(x)\big) = \Phi\Big(\sum_{k=1}^{m} w_k\, J_k(x)\Big) = \Phi\big(R_w(x)\big).$$

Since Φ is strictly decreasing, for any $x, x' \in D$ we have

$$R_w(x) \leq R_w(x') \quad \Longleftrightarrow \quad \Phi\big(R_w(x)\big) \geq \Phi\big(R_w(x')\big),$$

hence

$$\arg\min_{x\in D} R_w(x) = \arg\max_{x\in D} \mu_{\mathrm{FM}}(x).$$

Therefore the scalarized Multi-Criteria Risk Management problem is exactly the Fuzzy Multi-Criteria Risk Management problem with this choice of Ψ.

Combining (i) and (ii), we conclude that Fuzzy Multi-Criteria Risk Management generalizes both Fuzzy Risk Management and scalarized Multi-Criteria Risk Management. □

2.3|Neutrosophic Multi-Criteria Risk Management

Neutrosophic Multi-Criteria Risk Management models each criterion's risk by truth, indeterminacy, falsity degrees, combining them to rank uncertain decisions robustly.

Definition 2.9 (Neutrosophic Multi-Criteria Risk Management)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $D \subseteq \mathbb{R}^n$ be a nonempty set of admissible decisions. Let

$$K := \{1, \ldots, m\}$$

be a finite index set of risk criteria.

For each $k \in K$, let

$$L_k : D \longrightarrow L^{\infty}(\Omega, \mathcal{F}, P), \qquad x \longmapsto L_k(x)$$

denote the loss random variable associated with decision x under criterion k, and let

$$\rho_k : L^{\infty}(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a (coherent) risk measure for criterion k. Define the criterion-wise risk functionals

$$J_k : D \longrightarrow \mathbb{R}, \qquad J_k(x) := \rho_k\big(L_k(x)\big), \quad k = 1, \ldots, m,$$

and collect them into the risk vector

$$J : D \longrightarrow \mathbb{R}^m, \qquad J(x) := \big(J_1(x), \ldots, J_m(x)\big).$$

Let

$$\Phi_T,\ \Phi_I,\ \Phi_F : \mathbb{R}^m \longrightarrow [0,1]$$

be continuous functions, nonincreasing in each coordinate, generating respectively the *truth*, *indeterminacy*, and *falsity* degrees. For each $x \in D$ define the neutrosophic triple

$$T(x) := \Phi_T\big(J(x)\big), \qquad I(x) := \Phi_I\big(J(x)\big), \qquad F(x) := \Phi_F\big(J(x)\big).$$

The *neutrosophic multi-criteria decision set* is

$$\mathcal{N}_{\mathrm{MC}} := \big\{ (x,\ (T(x), I(x), F(x))) \mid x \in D \big\}.$$

Let $w_T, w_I, w_F \geq 0$ with $w_T + w_I + w_F = 1$ be fixed aggregation weights. Define the aggregated neutrosophic score

$$S_{\mathrm{MC}}(x) := w_T\, T(x) \ +\ w_I \left(1 - I(x)\right) \ +\ w_F \left(1 - F(x)\right), \qquad x \in D.$$

The *Neutrosophic Multi-Criteria Risk Management problem* is

$$x^* \in \arg\max_{x \in D}\ S_{\mathrm{MC}}(x).$$

The tuple

$$\mathsf{NMCRM} := \big(D, \{L_k\}_{k=1}^m, \{\rho_k\}_{k=1}^m, \Phi_T, \Phi_I, \Phi_F, w_T, w_I, w_F\big)$$

is called a *Neutrosophic Multi-Criteria Risk Management structure.*

Example 2.10 (Neutrosophic Multi-Criteria Risk Management for IT Vendor Selection)**.** Consider two IT vendors

$$D = \{x_A, x_B\},$$

and two risk criteria:

$$k = 1 :\ \text{financial loss (in million USD)}, \qquad k = 2 :\ \text{regulatory penalty score}.$$

Suppose that, after constructing loss random variables L_k and applying coherent risk measures ρ_k, the criterion-wise risks are

$$J_1(x_A) = 1.8, \quad J_2(x_A) = 1.4, \qquad J_1(x_B) = 2.1, \quad J_2(x_B) = 1.0.$$

Thus

$$J(x_A) = (1.8, 1.4), \qquad J(x_B) = (2.1, 1.0).$$

Let $(w_1, w_2) = (0.6, 0.4)$ and define the aggregated scalar risk

$$R(J_1, J_2) := w_1 J_1 + w_2 J_2.$$

Then

$$R(x_A) = 0.6 \cdot 1.8 + 0.4 \cdot 1.4 = 1.64, \qquad R(x_B) = 0.6 \cdot 2.1 + 0.4 \cdot 1.0 = 1.66.$$

Choose neutrosophic generators (with $M = 5$)

$$\Phi_T(J_1, J_2) := 1 - \frac{R(J_1, J_2)}{M}, \quad \Phi_I(J_1, J_2) := 0.2, \quad \Phi_F(J_1, J_2) := \frac{R(J_1, J_2)}{M},$$

so that for $x \in D$ we obtain

$$T(x) = \Phi_T(J(x)), \quad I(x) = \Phi_I(J(x)), \quad F(x) = \Phi_F(J(x)).$$

Hence

$$(T(x_A), I(x_A), F(x_A)) = \left(1 - \tfrac{1.64}{5},\ 0.2,\ \tfrac{1.64}{5}\right) = (0.672,\ 0.20,\ 0.328),$$
$$(T(x_B), I(x_B), F(x_B)) = \left(1 - \tfrac{1.66}{5},\ 0.2,\ \tfrac{1.66}{5}\right) = (0.668,\ 0.20,\ 0.332).$$

Take aggregation weights

$$w_T = 0.5, \quad w_I = 0.2, \quad w_F = 0.3 \qquad (w_T + w_I + w_F = 1),$$

and compute the neutrosophic scores

$$S_{\mathrm{MC}}(x) = w_T T(x) + w_I\big(1 - I(x)\big) + w_F\big(1 - F(x)\big).$$

Then

$$S_{\mathrm{MC}}(x_A) = 0.5 \cdot 0.672 + 0.2 \cdot 0.8 + 0.3 \cdot 0.672 = 0.6976,$$
$$S_{\mathrm{MC}}(x_B) = 0.5 \cdot 0.668 + 0.2 \cdot 0.8 + 0.3 \cdot 0.668 = 0.6944.$$

Since $S_{\mathrm{MC}}(x_A) > S_{\mathrm{MC}}(x_B)$, the Neutrosophic Multi-Criteria Risk Management framework selects vendor x_A as the preferred decision.

Theorem 2.11 (Neutrosophic MCRM generalizes both frameworks)**.** *The Neutrosophic Multi-Criteria Risk Management framework in Definition 2.9 simultaneously generalizes*

(i) the single-criterion Neutrosophic Risk Management model, and

(ii) the (crisp) scalarized Multi-Criteria Risk Management model.

Proof: (i) *Neutrosophic Risk Management as a special case.* Consider a single-criterion neutrosophic risk model with a single loss mapping $L : D \to L^\infty(\Omega, \mathcal{F}, P)$, a risk measure $\rho : L^\infty \to \mathbb{R}$, and scalar neutrosophic generators $\widehat{\Phi}_T, \widehat{\Phi}_I, \widehat{\Phi}_F : \mathbb{R} \to [0, 1]$, so that

$$T(x) = \widehat{\Phi}_T\big(\rho(L(x))\big), \quad I(x) = \widehat{\Phi}_I\big(\rho(L(x))\big), \quad F(x) = \widehat{\Phi}_F\big(\rho(L(x))\big),$$

and an aggregated score

$$S(x) = w_T T(x) + w_I(1 - I(x)) + w_F(1 - F(x)).$$

Embed this into Definition 2.9 by taking

$$m = 1, \quad L_1 := L, \quad \rho_1 := \rho,$$

so that $J_1(x) = \rho(L(x))$ and $J(x) = (J_1(x)) \in \mathbb{R}$. Define

$$\Phi_T(r) := \widehat{\Phi}_T(r), \quad \Phi_I(r) := \widehat{\Phi}_I(r), \quad \Phi_F(r) := \widehat{\Phi}_F(r), \qquad r \in \mathbb{R}.$$

Then for all $x \in D$ we have

$$T(x) = \Phi_T(J(x)) = \widehat{\Phi}_T(\rho(L(x))),$$

and analogously for $I(x)$ and $F(x)$; consequently $S_{\mathrm{MC}}(x) = S(x)$ and

$$\arg\max_{x \in D} S_{\mathrm{MC}}(x) = \arg\max_{x \in D} S(x).$$

Thus the single-criterion neutrosophic risk problem is recovered as the case $m = 1$ of Neutrosophic Multi-Criteria Risk Management.

(ii) *Scalarized Multi-Criteria Risk Management as a special case.* Consider a multi-criteria risk model with the same $D, \{L_k\}, \{\rho_k\}$ and risk vector $J(x)$, together with a weight vector $u = (u_1, \ldots, u_m)$, $u_k \geq 0$, $\sum_{k=1}^m u_k = 1$, and scalarized risk

$$R_u(x) := \sum_{k=1}^m u_k \, J_k(x).$$

The classical scalarized problem is $\min_{x \in D} R_u(x)$.

Choose a strictly decreasing bijection $\Phi : \mathbb{R} \to [0, 1]$ and set

$$\Phi_T(J_1, \ldots, J_m) := \Phi\Big(\sum_{k=1}^m u_k J_k\Big), \qquad \Phi_I \equiv 0, \qquad \Phi_F \equiv 0.$$

Take $w_T = 1$, $w_I = w_F = 0$. Then

$$T(x) = \Phi_T\big(J(x)\big) = \Phi\big(R_u(x)\big), \qquad I(x) = F(x) = 0,$$

and so

$$S_{\mathrm{MC}}(x) = w_T T(x) + w_I(1 - I(x)) + w_F(1 - F(x)) = \Phi\big(R_u(x)\big).$$

Because Φ is strictly decreasing, for any $x, x' \in D$,

$$R_u(x) \leq R_u(x') \quad \Longleftrightarrow \quad \Phi\big(R_u(x)\big) \geq \Phi\big(R_u(x')\big),$$

hence

$$\arg\min_{x\in D} R_u(x) = \arg\max_{x\in D} S_{\mathrm{MC}}(x).$$

Therefore the scalarized Multi-Criteria Risk Management problem is exactly the Neutrosophic Multi-Criteria Risk Management problem for this particular choice of $(\Phi_T, \Phi_I, \Phi_F, w_T, w_I, w_F)$.

Combining (i) and (ii), we conclude that Neutrosophic Multi-Criteria Risk Management generalizes both single-criterion Neutrosophic Risk Management and scalarized Multi-Criteria Risk Management. □

3|Conclusion

In this paper, we defined Multi-Criteria Risk Management, Fuzzy Multi-Criteria Risk Management, and Neutrosophic Multi-Criteria Risk Management, and we investigated their fundamental properties. In future work, we expect further extensions based on Plithogenic Sets[18, 19], Double-Valued Neutrosophic Sets [20], Uncertain Sets [21, 22], and Triple-Valued Neutrosophic Sets [23] to be actively explored.

Funding

No external funding or financial support was provided for this study.

Acknowledgments

The authors wish to thank all colleagues and mentors whose feedback and encouragement enriched this work. We are grateful to the community of researchers whose foundational contributions informed our developments. Special appreciation goes to the institutions that offered resources and technical infrastructure throughout this project.

Data Availability

This manuscript presents purely conceptual work without empirical data. Scholars interested in these ideas are invited to undertake experimental or case-study research to substantiate and extend the proposed frameworks.

Ethical Approval

This paper involves no human or animal subjects and thus did not require ethics committee review or approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors declare that there are no competing interests concerning the content or publication of this article.

Disclaimer

The theoretical models and propositions herein have not yet been subjected to practical validation. Readers should independently verify all citations and be aware that inadvertent inaccuracies may remain. The opinions expressed are those of the authors and do not necessarily represent the views of affiliated organizations.

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